MAT0206 - Real Analysis

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On the existence of $\sqrt{2}$

Let $S = \{x \in \mathbb{Q} : x \geq 0 \text{ and } x^2 < 2\}$.

a. Show that S is bounded above.

   Take $M = 3$. It is easy to verify M is an upper bound for S.

b. Let $c = \sup S$. Show that $c > 0$ and that $c^2 = 2$.

   Assume $c \leq 0$. Let then $x \in S, x = 1$, which implies that $x > c$: a contradiction with the upper bound property for the supremum. So, $c > 0$.

   Assume now that $c^2 > 2$. We can write c^2 as:

$$c^2 = 2 + \delta, \quad \delta \in \mathbb{R}, \delta > 0$$

   Let s' be defined as:

$$s'^2 = 2 + \frac{\delta}{2} \tag{1}$$

   From the definition of S we have:

$$x^2 < 2, \quad \forall x \in S \tag{2}$$

   We now subtract 1 from 2, obtaining:

$$\begin{aligned} x^2 - s'^2 &< 2 - \left(2 + \frac{\delta}{2}\right) \Rightarrow \\ x^2 - s'^2 &< 2 - 2 - \frac{\delta}{2} \Rightarrow \\ x^2 + \frac{\delta}{2} &< s'^2 \Rightarrow (\delta > 0) \\ x^2 &< s'^2 \Leftrightarrow (x, s' > 0) \\ x &< s', \quad \forall x \in S \end{aligned}$$

   Which contradicts the least upper bound property for the suppremum, which states:

$$\forall s, s < c \Rightarrow \exists x \in S : x > s,$$

which leads us to conclude that our assumption that $c^2 > 2$ is wrong. Let's assume then that $c^2 < 2$. We can then write c^2 as:

$$c^2 = 2 - \delta, \quad \delta \in \mathbb{R}, \delta > 0 \tag{3}$$

For each δ, we associate a number $\Lambda \in \mathbb{Q}$, such that:

$$\begin{cases} \text{if } \delta \in \mathbb{Q} & then \quad \Lambda = \frac{\delta}{2} \\ \text{if } \delta \notin \mathbb{Q} & then \quad \Lambda \text{ is any element [1] of the set } \left]0, \frac{\delta}{2}\right[\cap \mathbb{Q} \end{cases}$$

And let $x \in S$:

$$x^2 = 2 - \Lambda \tag{4}$$

Subtracting 4 from 3 we arrive at:

$$\begin{aligned} c^2 - x^2 &= 2 - \delta - (2 - \Lambda) \Rightarrow \\ c^2 - x^2 &= -\delta + \Lambda \Rightarrow \\ c^2 - x^2 &= \underbrace{\Lambda - \delta}_{<0} \Rightarrow \\ c^2 - x^2 &< 0 \Rightarrow c^2 < x^2 \Leftrightarrow (c, x, > 0) \\ c &< x \end{aligned}$$

So, $\exists x \in S : x > c$, which contradicts the fact that $c = \sup S$. $c^2 < 2$ is also false. So by the trichotomy property we arrive at $c^2 = 2$.

c. Conclude that S has no supremum in $\mathbb{Q}$ and that $\sqrt{2}$ exists.

$c^2 = 2$ has no roots in $\mathbb{Q}$, which implies that $c \notin \mathbb{Q}$. By the completeness axiom we conclude that $\sqrt{2}$, defined as $\sup S$, exists.

About

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[1] This set is always non-empty by the density of $\mathbb{Q}$ over $\mathbb{R}$