MAT0206 - Real Analysis

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List 1, Problem 14

Let A and B be two non-empty real subsets, such that $a \leq b$, $\forall a \in A$ and $\forall b \in B$. Prove that $\sup A \leq \inf B$. Using the same hypothesis, prove that $\sup A = \inf B$, if and only if, for any $\epsilon > 0$, $\exists a \in A, \exists b \in B : b - a < \epsilon$.

Solution

Let us begin with the first statement:

$$\sup A \leq \inf B$$

Assume by absurd that $\sup A > \inf B$. By the infimum property:

$$s > \inf B \Rightarrow \exists b \in B : b < s$$

Using this property: $\sup A > \inf B \Rightarrow \exists b \in B : b < \sup A$. But by the supremum property:

$$b < \sup A \Rightarrow \exists a \in A : a > b$$

Which contradicts our hypothesis that $a \leq b$, $\forall a \in A$, $\forall b \in B$, and so we conclude that $\sup A \leq \inf B$ must be true.

Let us consider now the second statement:

$$\sup A = \inf B \Leftrightarrow \forall \epsilon > 0, \exists a \in A, \exists b \in B : b - a < \epsilon.$$

$(\Leftarrow)$

We will first consider the left implication. We know by hypothesis that:

$$\forall \epsilon > 0, \exists a \in A, \exists b \in B : b - a < \epsilon$$

By the previous result we proved, we have that: $\sup A \leq \inf B$. So the negative of $\sup A = \inf B$ is $\sup A > \inf B$. Assume by absurd the latter statement ($\sup A > \inf B$). That implies that $\exists \delta, \delta > 0 : \inf B - \sup A = \delta$. By our hypothesis and taking $\epsilon = \frac{\delta}{2}$, we have that:

$$\begin{aligned}
\exists a \in A, \exists b \in B : b - a &< \frac{\delta}{2} \Rightarrow (\text{as } \inf B \leq b)\\
\inf B - a &< \frac{\delta}{2} \Rightarrow (\text{as } \sup A \geq a)\\
\inf B - \sup A &< \frac{\delta}{2}
\end{aligned}$$

But we assumed that $\inf B - \sup A = \delta$, $\delta > 0$, and so we arrived at a contradiction. Thus $\sup A = \inf B$ and we proved the left implication.

$(\Rightarrow)$

We will now show the right implication. Consider first the following lemma:

Lemma 1. *Let A, and B be two non-empty real subsets, and ϵ_0 a real number, $\epsilon_0 > 0$. It is always true that:*

$$\forall a \in A, \forall b \in B, \quad b - a \geq \epsilon_0 \quad \Rightarrow \quad b - \sup A \geq \epsilon_0$$

Demonstration We know by hypothesis that

$$\forall a \in A, \forall b \in B, \quad b - a \geq \epsilon_0,$$

and by the supremum property, we have:

$$\forall \epsilon, \epsilon > 0, \exists a \in A : a > \sup A + \epsilon$$

Taking $\epsilon = \epsilon_0$, we know by the above property that there exists an a in A such that $a > \sup A + \epsilon_0$. In particular, by our hypothesis, this implies that:

$$\begin{aligned}
\forall b \in B : b - (\sup A + \epsilon_0) \geq \epsilon_0 \quad &\Rightarrow\\
b - \sup A \geq 2\epsilon_0 \quad &\Rightarrow \quad b - \sup A \geq \epsilon_0,
\end{aligned}$$

which is where we wanted to arrive. ■

Our hypothesis (p) states that:

$$\sup A = \inf B$$

Let us consider $\neg q$ and show that $(p \wedge \neg q) \Rightarrow c$. $\neg q$ states:

$$\exists \epsilon_0, \epsilon_0 > 0 : \forall a \in A, \forall b \in B, \quad b - a \geq \epsilon_0$$

Using Lemma 1 and $\neg q$ we have that:

$$\forall b \in B, \quad b - \sup A \geq \epsilon_0$$

But by p (our hypothesis), $\sup A = \inf B$, and so:

$$\forall b \in B, \quad b - \inf B \geq \epsilon_0 \Rightarrow \inf B + \epsilon_0 \leq b,$$

which is a contradiction with the greatest lower bound property of the infimum. So as $(p \wedge \neg q) \Rightarrow c$, $p \Rightarrow q$ and we proved the right implication.

About

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